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A hypergeometric system of the Heun equation and middle convolution

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Abstract
A new method of obtaining integral transformation of the Heun equation is described. The Heun equation is reduced to the so-called hypergeometric system, a linear Fuchsian system of rank 3, and a Detweiler and Reiter algebraic analogue of the Katz middle convolution functor is applied. A linear Fuchsian system of rank 2 associated with the Heun equation is also studied.

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1. Introduction

1.1. The Heun equation

The Heun equation [12, p 292] is given by

$$t(t-1)(t-\lambda)y'' + \{\gamma(t-1)(t-\lambda) + \delta t(t-\lambda) + \epsilon t(t-1)\}y' + \alpha\beta(t-a)y = 0, \tag{1}$$

where $' = \mathrm{d}/\mathrm{d}t$. This equation has four regular singular points in $\mathbb{P}^1$ and, hence, is Fuchsian. The Riemann scheme is given by

$$\wp\left\{\begin{matrix} 0 & 1 & \lambda & \infty & \\ 0 & 0 & 0 & \alpha & t \\ 1-\gamma & 1-\delta & 1-\epsilon & \beta & \end{matrix}\right\},$$

where the Fuchs relation

$$\alpha + \beta - \gamma - \delta - \epsilon + 1 = 0 \tag{2}$$

holds. The Heun equation includes only one accessory parameter a, which cannot be determined by the characteristic exponents.

If we differentiate the Heun equation (1) with respect to t, we have a third-order linear differential equation:

$$t(t-1)(t-\lambda)y''' + \{(\gamma+1)(t-1)(t-\lambda) + (\delta+1)t(t-\lambda) + (\epsilon+1)t(t-1)\}y'' \\ + \{(\gamma+\epsilon)(t-1) + (\gamma+\delta)(t-\lambda) + (\delta+\epsilon)t + \alpha\beta(t-a)\}y' + \alpha\beta y = 0, \quad (3)$$

for which the Riemann scheme is given by

$$\wp\left\{\begin{matrix} 0 & 1 & \lambda & \infty & \\ 0 & 0 & 0 & 1 & \\ 1 & 1 & 1 & \alpha & t \\ 1-\gamma & 1-\delta & 1-\epsilon & \beta & \end{matrix}\right\}.$$

This shows that the behavior of solutions near regular singular points does not change and one holomorphic solution is added to form a fundamental set of solutions near each regular singular point.

Heun's equation and its confluent cases appear in many areas of mathematical physics. For instance, the classical problem of handling the Hemholtz or Laplace equation in all but the simplest coordinate systems leads to separated differential equations (Mathieu's, Ince's, Lamé's, the spheroidal wave equation) which turn out to be of the Heun class [16]. Other examples in physics and mathematics include the irradiation-amplified diffusion in crystal, the line-tension model [17], nonlinear elastic polymers in a random flow [13] and the eigenvalue problem of the scalar Laplacian for the toric Sasaki–Einstein manifolds [15].

Some properties of the Heun equation are discussed in [16, 17]. Recently, the integral transformations of the Fuchsian differential equations have attracted a lot of attention [7, 8, 11, 14, 18], where the Heun equation and the sixth Painlevé equation are studied in detail. In this paper, we explain a new approach to study the Heun equation by using the Katz middle convolution functor for the Heun hypergeometric system.

1.2. Hypergeometric system of the Heun equation

It is shown in [12, chapter 4] that a Fuchsian linear differential equation of order n can be reduced to a system of the form

$$(t-B)\frac{\mathrm{d}Y}{\mathrm{d}t} = AY,$$

where B is a diagonal matrix, A is a constant matrix and $Y = (y_1, \ldots, y_n)^T$. Such systems are called *hypergeometric systems*.

In particular, the reduction problem for the Heun equation is solved as follows [12, p 296]. Taking

$$y_1 = y, \qquad y_2 = ty_1' + (\gamma-1)y_1,$$
$$\lambda y_3 = (a\alpha\beta + \epsilon(1-\gamma))y_1 + ((t-1)\gamma + t\delta)\lambda y_1' + (t-1)t\lambda y_1'',$$

the required hypergeometric system is given by

$$\begin{pmatrix} t & 0 & 0 \\ 0 & t-1 & 0 \\ 0 & 0 & t-\lambda \end{pmatrix}\begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} 1-\gamma & 1 & 0 \\ \alpha_{21} & -\delta & 1 \\ \alpha_{31} & \alpha_{32} & -\epsilon-1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \quad (4)$$

where

$$\alpha_{21} = (\gamma-1)\delta - g_0, \qquad \alpha_{31} = \alpha\beta(\gamma-2) + (1-\gamma)\epsilon(\delta+\gamma-2) + (\epsilon-\gamma+2)g_0,$$

$$\alpha_{32} = \epsilon(\delta+\gamma-2) - \alpha\beta + g_0, \qquad g_0 = \frac{1}{\lambda}\{\epsilon(1-\gamma) + \alpha\beta a\}.$$

Eliminating y_2 and y_3 from (4), we get a scalar differential equation of order 3 for the function $y_1 = y$ which coincides with equation (3).

Each diagonal element of matrix A in (4) is equal to the characteristic exponent of (3) at the respective singular point modulo integers. The eigenvalues of $-A$ are equal to the characteristic exponents at infinity of equation (3).

1.3. Outline of results

In section 2, we explain an operation on Fuchsian systems called middle convolution. It is applied to the hypergeometric system of the Heun equation in section 3. The main result is given by the theorem which relates two Heun equations with specific values of the parameters via middle convolution. Further, we study rank 2 Fuchsian systems associated via middle convolution with the Heun equation. Finally, a conclusion with some more comments rounds off the paper.

2. Middle convolution

In this section, we discuss the Dettweiler and Reiter algebraic construction of the Katz middle convolution functor following [2–4] and its significance. An algorithm and a relation of middle convolution to a specific integral transformation are briefly outlined following [5].

Katz described all irreducible and physically rigid systems on the punctured affine line in [10] and introduced a middle convolution functor mc_μ in the category of perverse sheaves which preserves important properties of local systems such as the number of singularities, an index of rigidity and irreducibility, but in general changes the rank and the monodromy group. By Katz's theorem, one can obtain any irreducible rigid local system on the punctured affine line from a local system of rank 1 by applying a suitable sequence of middle convolutions and scalar multiplications.

Recently, Dettweiler and Reiter gave a purely algebraic analogue of the Katz middle convolution functor in [2–4]. They presented an algorithm which allows one to construct Fuchsian systems corresponding under the Riemann–Hilbert correspondence to irreducible rigid local systems. The main approach in [2, 3] was the generalization of normal forms of the Pochhammer equation. Dettweiler and Reiter not only reproduced Katz's main result, but also presented both the multiplicative and the additive versions of their algebraic analogue, studied their main properties, gave a cohomological interpretation and applied their theory to construct explicit algebraic solutions to the sixth Painlevé equation in [4].

The Dettweiler and Reiter algorithm can formally be applied to any Fuchsian system, and not necessarily the rigid one. It guarantees that the resulting linear system is irreducible and has the same number of singularities as the initial Fuchsian system. However, the dimension of matrices of the resulting Fuchsian system may change, and hence the rank of the Fuchsian system may be different. In [7] the algorithm was applied to the Fuchsian system of rank 2 with four singularities, and Okamoto's birational transformation (Bäcklund transformation) was rediscovered for the deformation equation, which is the sixth Painlevé equation in this case. Another approach to this result is given in [14] where an integral transformation is applied directly to a scalar Fuchsian differential equation. Although Fuchsian systems before and after middle convolution are related by a certain integral transformation along a Pochhammer contour (see below for more details), the algebraic interpretation of middle convolution is more efficient for some problems than the direct method for scalar Fuchsian equations. For instance, by using the method of middle convolution one can easily construct Fuchsian systems of an arbitrary rank equivalent to a given system (or having the same family of deformation equations [8]) which is more involved by other direct methods. Furthermore, middle convolution and addition are operations on Fuchsian systems of differential equations which preserve a number of accessory parameters. It was shown in [8] that these operations also preserve deformation equations.

The multiplicative version of the middle convolution functor denoted by MC_λ is a functor of the category of finite dimensional $\mathbb{C}[F_r]$-modules of the free group F_r on r generators to itself (local systems), where $\lambda \in \mathbb{C}^\times$ is a parameter. It is a transformation sending r matrices in $GL_n(\mathbb{C})$ to other r matrices in $GL_m(\mathbb{C})$, where usually m is not equal to n. Up to a simultaneous conjugation in $GL_m(\mathbb{C})$, this transformation commutes with the Artin braid group [3]. There exists a parallel functor in the category of the Fuchsian systems, mc_μ, which is related to MC_λ via the Riemann–Hilbert correspondence by a monodromy map.

The construction of mc_μ is as follows. Let $\mathbf{A} = (A_1, \ldots, A_r)$, $A_k \in \mathbb{C}^{n\times n}$. Let us also fix points $t = t_k \in \mathbb{C}$, $k = 1, \ldots, r$, and consider a Fuchsian system of rank n given by

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = \sum_{k=1}^{r} \frac{A_k}{t - t_k} Y. \tag{5}$$

First, the operation of *addition* is simply a change of the eigenvalues of the residue matrix: $A_k \to A_k + aI_n$, where $a \in \mathbb{C}$ and I_n is the identity matrix.

For $\mu \in \mathbb{C}$, one defines *convolution matrices* $\mathbf{B} = c_\mu(\mathbf{A}) = (B_1, \ldots, B_r)$ by

$$B_k = \begin{pmatrix} 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ A_1 & \ldots & A_{k-1} & A_k + \mu & A_{k+1} & \ldots & A_r \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \end{pmatrix} \in \mathbb{C}^{nr\times nr} \tag{6}$$

such that B_k is zero outside the kth block row.

The convolution matrices define a new Fuchsian system of rank nr with the same number of singularities as in the original Fuchsian system:

$$\frac{\mathrm{d}Y_1}{\mathrm{d}t} = \sum_{k=1}^{r} \frac{B_k}{t - t_k} Y_1. \tag{7}$$

This system may be reducible. In general, there are the following invariant subspaces of the column vector space $\mathbb{C}^{nr}$:

$$\mathcal{L}_k = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mathrm{Ker}(A_k) \\ 0 \\ \vdots \\ 0 \end{pmatrix} (k\text{-th entry}), \qquad k = 1, \ldots, r, \tag{8}$$

and

$$\mathcal{K} = \bigcap_{k=1}^{r} \mathrm{Ker}(B_k) = \mathrm{Ker}(B_1 + \cdots + B_r). \tag{9}$$

Let us denote $\mathcal{L} = \oplus_{k=1}^{r} \mathcal{L}_k$ and fix an isomorphism between $\mathbb{C}^{nr}/(\mathcal{K}+\mathcal{L})$ and $\mathbb{C}^m$ for some m. The matrices $\widetilde{\mathbf{B}} = mc_\mu(\mathbf{A}) := (\widetilde{B}_1, \ldots, \widetilde{B}_r) \in \mathbb{C}^{m\times m}$, where $\widetilde{B}_k$ is induced by the action of B_k on $\mathbb{C}^m \simeq \mathbb{C}^{nr}/(\mathcal{K}+\mathcal{L})$, are called *the additive version of the middle convolution of* $\mathbf{A}$ *with parameter* μ. Thus, the resulting irreducible Fuchsian system of rank m is given by

$$\frac{\mathrm{d}Y_2}{\mathrm{d}t} = \sum_{k=1}^{r} \frac{\widetilde{B}_k}{t-t_k} Y_2. \tag{10}$$

To sum up, the additive version of middle convolution depends on a scalar $\mu \in \mathbb{C}$ and is denoted by mc_μ. It is a transformation on tuples of matrices:

$$(A_1, \ldots, A_r) \in (\mathbb{C}^{n\times n}))^r \to mc_\mu(A_1, \ldots, A_r) = (\widetilde{B}_1, \ldots, \widetilde{B}_r) \in (\mathbb{C}^{m\times m}))^r.$$

Finally, we describe the relation between the convolution operation c_μ and integral transformations following [5] for completeness.

Let $g := (g_{i,j})$ be a matrix with entries $g_{i,j}$ such that they are (multi-valued) holomorphic functions on $X := \mathbb{C}\setminus T, T := \{t_1, \ldots, t_r\} \subset \mathbb{C}, t_i \neq t_j$ for $i \neq j$. Assume that the path α_{r+1} encircles an open neighborhood U of y_0 and the path α_i encircles the point t_i. Then the matrix-valued function

$$I^{\mu}_{[\alpha_{r+1},\alpha_i]}(g)(y) := \int_{[\alpha_{r+1},\alpha_i]} g(x)(y-x)^{\mu-1}\,\mathrm{d}x,\ y \in U,$$

is called the *Euler transform* of g with respect to the Pochhamer contour $[\alpha_{r+1}, \alpha_i] := \alpha_{r+1}^{-1}\alpha_i^{-1}\alpha_{r+1}\alpha_i$ and the parameter $\mu \in \mathbb{C}$.

Let $\mathbf{A} := (A_1, \ldots, A_r), A_i \in \mathbb{C}^{n\times n}$ be the residue matrices of the Fuchsian system (5) and $F(t)$ be its fundamental solution. Denote

$$G(t) := \begin{pmatrix} F(t)(t-t_1)^{-1} \\ \vdots \\ F(t)(t-t_r)^{-1} \end{pmatrix}$$

and introduce the period matrix

$$I^{\mu}(y) := \left(I^{\mu}_{[\alpha_{r+1},\alpha_1]}(G)(y), \ldots, I^{\mu}_{[\alpha_{r+1},\alpha_r]}(G)(y)\right).$$

Then according to [5] the columns of the period matrix $I^\mu(y)$ are solutions to the Fuchsian system (7) obtained by the convolution with parameter $\mu-1$, i.e. $c_{\mu-1}(\mathbf{A})$, where y is contained in a small open neighborhood U of y_0 (which is encircled by α_{r+1}).

3. Middle convolution and the hypergeometric system of the Heun equation

In this section, we apply the Dettweiler and Reiter algorithm of middle convolution to the hypergeometric system (4). In the result, we obtain a new hypergeometric system of the Heun equation with new values of the parameters provided we have a special choice of parameter μ in middle convolution. The new parameters of the hypergeometric system coincide with the parameters given in [11] where the integral transformation was applied directly to the Heun equation. The hypergeometric systems seem to be more convenient and natural to study special functions such as the Heun function in comparison with the systems studied in [18] via middle convolution and using the space of initial conditions of the sixth Painlevé equation. We also conjecture that the hypergeometric systems associated with the Garnier systems would be useful to calculate easily birational symmetries.

The hypergeometric system of the Heun equation (4) is of rank $n = 3$ with three finite singularities $t = 0, 1, \lambda$ and a singularity at $t = \infty$. Clearly, it is of form (5) with $r = 3$

by construction. The nonzero eigenvalues of the residue matrices A_k are $1-\gamma, -\delta, -1-\epsilon$ respectively. The eigenvalues of the residue matrix at $t=\infty$ are $1, \alpha, \beta$.

We apply the middle convolution algorithm with parameter μ being not equal to the eigenvalues of the residue matrix at $t=\infty$. Also assume that $\mu \neq 0$. In this case, the convolution matrices are of dimension $nr = 9$. The corresponding system (7) is reducible. The invariant subspace $\mathcal{K}$ is empty and the subspace $\mathcal{L}$ is spanned by six vectors. The quotient space $\mathbb{C}^3 \simeq \mathbb{C}^9/(\mathcal{K}+\mathcal{L})$ is constructed by adding three more vectors to the basis of the invariant subspaces, for instance vectors e_1, e_6, e_7 where e_k has 1 at position k and other elements are equal to zero.

The matrices $\widetilde{B}_k$ of the resulting system (10) have a nonzero kth row and are given by

$$\widetilde{B}_1 = \begin{pmatrix} 1-\gamma+\mu & 1/(1-\gamma) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\widetilde{B}_2 = \begin{pmatrix} 0 & 0 & 0 \\ \frac{(1-\gamma)(-(a\alpha\beta)+(\gamma-1)(\epsilon+\delta\lambda))}{\lambda} & \mu-\delta & b_2 + \frac{(2-\gamma+\epsilon)(a\alpha\beta+\epsilon-\gamma\epsilon)}{\lambda} \\ 0 & 0 & 0 \end{pmatrix},$$

$$\widetilde{B}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1-\gamma & \frac{a\alpha\beta+\epsilon-\gamma\epsilon-((\alpha-\epsilon)(\beta-\epsilon)+\epsilon)\lambda}{(2-\gamma+\epsilon)(a\alpha\beta+\epsilon-\gamma\epsilon)+(\alpha\beta(-2+\gamma)-(-1+\gamma)(-2+\gamma+\delta)\epsilon)\lambda} & \mu-1-\epsilon \end{pmatrix},$$

where $b_2 = \alpha\beta\,(\gamma-2)-(\gamma-1)\,(\gamma+\delta-2)\,\epsilon$. The nonzero eigenvalues of these matrices are $1-\gamma+\mu, \mu-\delta$ and $\mu-1-\epsilon$ respectively. The eigenvalues of the new residue matrix at $t=\infty$ of (10) are $1-\mu, \alpha-\mu, \beta-\mu$.

Next, we make a transformation $Y_2 = DY_3$ in (10) with an appropriate matrix $D \in GL(3,\mathbb{C})$ such that the system for Y_3 has a form similar to (4). We can calculate that provided $\gamma \neq 1$, the matrix D is diagonal with diagonal elements $D_{11} = 1/(1-\gamma)$, $D_{22} = 1$ and $D_{33}(\widetilde{B}_2)_{23} = 1$.

Thus, we have a new hypergeometric system of the Heun equation with new values of the parameters $\alpha_1, \beta_1, \gamma_1, \delta_1, \epsilon_1$ ($\epsilon_1 = 1+\alpha_1+\beta_1-\gamma_1-\delta_1$) in the result of middle convolution applied to system (4). However, this holds only for special values of parameter μ of middle convolution.

Theorem. *The hypergeometric system of the Heun equation (4) is related via middle convolution to parameter μ and a gauge transformation to another hypergeometric system (4) with new values of the parameters being given by either*

$$\mu = \alpha-1,\ (2-\alpha-\alpha_1)(1-\alpha+\beta-\alpha_1) = 0, \qquad \alpha_1\beta_1 = (\alpha-2)(\alpha-\beta-1),$$

$$\gamma_1 = \gamma-\alpha+1,\ \delta_1 = \delta-\alpha+1,\ \epsilon_1 = \epsilon-\alpha+1,$$

$$a_1 = \frac{1+\beta-\delta+\alpha(\delta-1+\beta(a-1))+(\alpha-1)(\alpha-\gamma-\delta)\lambda}{(\alpha-2)(\alpha-\beta-1)}$$

or

$$\mu = \beta-1,\ (2-\beta-\alpha_1)(1+\alpha-\beta-\alpha_1) = 0, \qquad \alpha_1\beta_1 = -(\beta-2)(\alpha-\beta+1),$$

$$\gamma_1 = \gamma-\beta+1,\ \delta_1 = \delta-\beta+1,\ \epsilon_1 = \epsilon-\beta+1,$$

$$a_1 = -\frac{\alpha(1+(a-1)\beta)+(\beta-1)(\delta-1+(\beta-\gamma-\delta)\lambda)}{(\beta-2)(\alpha-\beta+1)}.$$

Note that in the case $\gamma = \alpha - \delta$ (or $\gamma = \beta - \delta$), the first (or second) formula for a_1 in the theorem does not depend on λ.

We observe that the residue matrices A_k and the matrices after middle convolution and gauge transformation differ by μE_{kk}, where E_{kk} has 1 at position (k, k) and all other elements zero. This also follows from the general properties of the Euler transform applied to Okubo-type systems [12].

The parameters in the theorem coincide with parameters given in [11, 18] where the integral transformation of the Heun equation is studied by different methods. Therefore, we have given an algebraic interpretation (on the level of the hypergeometric system) of the integral transformation of the Heun equation.

4. The Heun equation and Fuchsian differential equations with apparent singularities

In this section, we show that function y_2 in (4) satisfies the second-order linear Fuchsian equation with singularities $t = 0, 1, \lambda, \infty$ and one apparent singularity which is parametrized by the accessory parameter of the Heun equation. Isomonodromic deformations lead to the constraint for the parameters, which, in turn, leads to the confluence of the apparent singularity with the point at infinity. Hence, we have Heun's equation in the limit. Such confluences were studied in detail in [18] where the relation to the space of initial conditions of the sixth Painlevé equation is discussed. In the following section, we shall study middle convolution with special values of the parameter such that the resulting system (10) is of rank 2 ($m = 2$). In this case, a scalar equation for the first (or second) component of vector Y_2 in (10) satisfies the second-order linear Fuchsian equation with one apparent singularity as well.

Recall that a scalar linear differential equation with rational coefficients is called Fuchsian if it has only regular singularities. An apparent singularity means that the solution is analytic at this point [9, chapter 3].

The second-order Fuchsian differential equation with singularities $t = 0, 1, \lambda, \infty$ and apparent singularity $t = L$ can be written in the following form [9, p 169]:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \left(\frac{1-\theta_0}{t} + \frac{1-\theta_1}{t-1} + \frac{1-\theta_2}{t-\lambda} - \frac{1}{t-L}\right)\frac{\mathrm{d}y}{\mathrm{d}t} + \left(\frac{k_1(k_2+1)}{t(t-1)} + \frac{L(L-1)M}{t(t-1)(t-L)} - \frac{\lambda(\lambda-1)H}{t(t-1)(t-\lambda)}\right) y = 0. \tag{11}$$

The monodromy representation of this differential equation is a homomorphism, defined up to a conjugation, from the fundamental group of the punctured plane $\mathbb{C} \setminus \{0, 1, \lambda\}$ to $GL(2, \mathbb{C})$. Monodromy matrices do not depend on λ if $L = L(\lambda)$ satisfies the sixth Painlevé equation (a second-order nonlinear differential equation, the solutions of which have no movable critical points). Equivalently, the monodromy does not depend on λ if functions $L(\lambda)$ and $M = M(\lambda)$ satisfy the following Hamiltonian system:

$$\frac{\mathrm{d}L}{\mathrm{d}\lambda} = \frac{\partial H}{\partial M}, \qquad \frac{\mathrm{d}M}{\mathrm{d}\lambda} = -\frac{\partial H}{\partial L}, \tag{12}$$

where the Hamiltonian is given by

$$H = \frac{1}{\lambda(\lambda-1)}\{L(L-1)(L-\lambda)M^2 - (\theta_0(L-1)(L-\lambda) + \theta_1 L(L-\lambda) + (\theta_2 - 1)L(L-1))M + k_1(k_2+1)(L-\lambda)\}$$

with $\theta_0 + \theta_1 + \theta_2 + k_1 + k_2 = 0$. Thus, the monodromy preserving deformation of linear equation (11) is governed by the Hamiltonian system (12). We remark that eliminating $M(\lambda)$ between these equations yields the sixth Painlevé equation for the function $L(\lambda)$ [9].

Let us first illustrate how to use equation (11) when a given differential transformation is applied to the Heun equation.

Assume that the function $v(t)$ satisfies the following Heun equation:

$$v'' + \left(\frac{\gamma_1}{t} + \frac{\delta_1}{t-1} + \frac{\epsilon_1}{t-\lambda}\right) v' + \frac{\alpha_1\beta_1 t - q_1}{t(t-1)(t-\lambda)} v = 0$$

with $\epsilon_1 = 1 + \alpha_1 + \beta_1 - \gamma_1 - \delta_1$. Then the function $y(t) = v'(t)$ satisfies equation (11) with $\theta_0 = -\gamma_1, \theta_1 = -\delta_1, \theta_2 = -1 - \alpha_1 - \beta_1 + \gamma_1 + \delta_1$,

$$L = \frac{q_1}{\alpha_1\beta_1}, \qquad M = \frac{-L^2(1+\alpha_1+\beta_1) - \lambda\gamma_1 + L(1+\alpha_1+\beta_1+\lambda\gamma_1-\delta_1+\lambda\delta_1)}{L(L-1)(L-\lambda)}$$

and either $k_1 = \alpha_1 + 1, k_2 = \beta_1$ or $k_1 = \beta_1 + 1, k_2 = \alpha_1$. Substituting L and M as functions of λ into system (12) yields a constraint $\alpha_1\beta_1 = 0$. Let us take $\alpha_1 = 0$. Then the apparent singularity becomes infinity and we get the fact that the function y satisfies equation with four singularities, which is Heun's equation, given by

$$y'' + \left(\frac{\gamma}{t} + \frac{\delta}{t-1} + \frac{\epsilon}{t-\lambda}\right) y' + \frac{\alpha\beta t - q}{t(t-1)(t-\lambda)} y = 0$$

with $\gamma = \gamma_1+1, \delta = \delta_1+1, \epsilon = 2+\beta_1-\gamma_1-\delta_1, \beta_1 = -1+\alpha\beta/2, q_1 = q-\alpha\beta/2+\delta_1-\gamma_1\lambda-\delta_1\lambda$. The condition $\epsilon = 1 + \alpha + \beta - \gamma - \delta$ should also be satisfied which leads to a constraint on α and β of the form $(\alpha - 2)(\beta - 2) = 0$. We note that transformation $y(t) = v'(t)$ with $\alpha = 2$ or $\beta = 2$ is contained in proposition 6.3 in [18]. However, we have derived it naturally by using equation (11). In general, if $y(t)$ satisfy a linear differential equation of the form

$$g_2(t)y'' + g_1(t)y' + g_0(t)y = 0,$$

then the function $y_1 = h_0(t)y' + h_1(t)y$ satisfies a linear differential equation

$$p_2(t)y_1'' + p_1(t)y_1' + p_0(t)y_1 = 0,$$

with $p_2(t) = g_2(g_1h_0h_1 - g_0h_0^2 - g_2(h_1^2 + h_1h_0' - h_0h_1'))$. So, it is possible that new singularities are added. By choosing the functions h_0 and h_1 appropriately, we can add only apparent singularities. We can also study transformations $y_1 = h_0(t)y^{(k)} + \cdots + h_k(t)y, k \in \mathbb{N}$, in a similar way.

Next, we study function y_2 of the hypergeometric system (4). It satisfies the second-order linear differential equation with the following Riemann scheme:

$$\wp \left\{ \begin{matrix} 0 & 1 & \lambda & L & \infty & \\ 0 & 0 & 0 & 0 & \alpha & t \\ 2-\gamma & -\delta & -\epsilon & 2 & \beta & \end{matrix} \right\},$$

where

$$L = \frac{a\alpha\beta - (\gamma-1)(\epsilon+\delta\lambda)}{\alpha\beta - (\gamma-1)(\epsilon+\delta)}$$

is an apparent singularity. Thus, we have equation (11) with

$$\theta_0 = 2-\gamma, \qquad \theta_1 = -\delta, \qquad \theta_2 = -\epsilon,$$

$$M = -\frac{\alpha\beta(\alpha\beta - (\gamma-1)(\delta+\epsilon))(a(\delta+\epsilon) - \epsilon - \delta\lambda)}{(a\alpha\beta + (\gamma-1)\epsilon(\lambda-1) - \alpha\beta\lambda)((a-1)\alpha\beta + \delta(\gamma-1+\lambda(1-\gamma)))}$$

and either $k_1 = \beta, k_2 = \alpha - 1$ or $k_1 = \alpha, k_2 = \beta - 1$. So the functions L and M are parametrized by the accessory parameter a of the Heun equation. Eliminating a between L and M yields

$$M = \frac{L(\epsilon+\delta) - \epsilon - \delta\lambda}{(\lambda - L)(L-1)}.$$

In this case, the Hamiltonian system (12) is satisfied provided $(1+\alpha-\gamma)(1+\beta-\gamma)=0$. Hence, $L=\infty$ and we can reduce equation (11) to the Heun equation. We can also study the space of initial conditions for the sixth Painlevé equation as in [18]. Similarly, the function $y_3(t)$ of (4) satisfies the second-order linear differential equation with another apparent singularity.

5. Middle convolution and rank 2 Fuchsian systems

In this section, middle convolution with special values of the parameter is applied to the Heun hypergeometric system. We choose the parameter of middle convolution equal to one of the eigenvalues of the residue matrix at infinity of system (4), i.e. we compute mc_1 and mc_α (the case mc_β is by analogy). In these cases $\mathbb{C}^2 \simeq \mathbb{C}^9/(\mathcal{K}+\mathcal{L})$ because both invariant subspaces are nonempty, so the resulting Fuchsian system (10) is of rank 2. It is not of type (4) and, therefore, the aim of this section is to reduce system (10) to the Heun equation by using a scalar Fuchsian equation of second order with an apparent singularity (11). For the general theory of Fuchsian differential systems and scalar equations, see, for instance, [9, 1, 6].

5.1. mc₁

First, let us consider the case when $\gamma=2$ and apply mc_1. By adding the vectors e_1+e_6 and e_7 to the basis of $\mathcal{K}+\mathcal{L}$, we get system (10) with

$$\widetilde{B}_1=\begin{pmatrix}0 & 0\\ -1/\epsilon & 0\end{pmatrix},\qquad \widetilde{B}_3=\begin{pmatrix}0 & 0\\ \frac{(a\alpha\beta-\epsilon)(\epsilon-1)+\lambda(\alpha\beta-2\delta\epsilon)}{\epsilon(\epsilon-a\alpha\beta+\delta\lambda)} & -\epsilon\end{pmatrix},$$

$$\widetilde{B}_2=\begin{pmatrix}\frac{a\alpha\beta-\epsilon+\lambda(1-2\delta)}{\lambda} & \frac{-\epsilon(\epsilon-a\alpha\beta+\delta\lambda)}{\lambda}\\ \frac{\epsilon-a\alpha\beta+\lambda(2\delta-1)}{\epsilon\lambda} & \frac{-a\alpha\beta+\epsilon+\delta\lambda}{\lambda}\end{pmatrix}.$$

Thus, when $\gamma=2$, the equation for the first component of vector Y_2 is the Heun equation with new values of the parameters $\alpha_1,\beta_1,\gamma_1=0,\delta_1=\delta,\epsilon_1=\alpha+\beta-1-\delta$ and $a_1=(1-\alpha-\beta+a\alpha\beta+\delta(1-\lambda))/(\alpha_1\beta_1), (\alpha-1-\alpha_1)(1-\beta+\alpha_1)=0, \alpha_1\beta_1=(\alpha-1)(\beta-1)$.

In general, applying mc_1 to system (4) and adding vectors e_1, e_7 to the basis of the invariant subspaces yield system (10) with

$$\widetilde{B}_1=\begin{pmatrix}2-\gamma & 0\\ 0 & 0\end{pmatrix},\qquad \widetilde{B}_3=\begin{pmatrix}0 & 0\\ 1-\gamma & -\epsilon\end{pmatrix},$$

$$\widetilde{B}_2=\begin{pmatrix}\frac{a\alpha\beta-(\gamma-1)(\epsilon+\delta\lambda)}{(\gamma-2)\lambda} & b_{12}\\ b_{21} & \frac{-a\alpha\beta+\epsilon(\gamma-\lambda-1)+(\alpha+\beta-1)\lambda}{(\gamma-2)\lambda},\end{pmatrix},$$

where b_{12} and b_{21} are complicated expressions in the parameters of the Heun equation.

The eigenvalues of the matrices $\widetilde{B}_k, k=1,2,3$, are as follows: $2-\gamma, 0; 1-\delta, 0; -\epsilon, 0$. The residue matrix at infinity is the matrix $-(\widetilde{B}_1+\widetilde{B}_2+\widetilde{B}_3)$ with eigenvalues $\alpha-1, \beta-1$. Next, we diagonalize this matrix by using a gauge transformation for the rank 2 system and write down the scalar equation for the first component in (10). It is of form (11) with $\theta_0=2-\gamma, \theta_1=1-\delta, \theta_2=-1-\alpha-\beta+\gamma+\delta$ and $k_1(1+k_2)=(\alpha-1)\beta$. The apparent singularity L is parametrized by the parameters of the Heun equation and by the accessory parameter a. Eliminating a between L and M as in the previous section yields $M=\epsilon/(\lambda-L)+(2+\beta-\gamma-\delta)/(L-1)$. Substituting L and M into (12) yields condition $(1+\beta-\gamma)(2+\beta-\gamma-\delta)=0$ which leads to $L=1$. Hence, the apparent singularity coalesces with singularity $t = 1$ and we can choose parameters such that the equation becomes Heun's equation (1). For instance, if $\delta = 2 + \beta - \gamma$, then $\alpha_1 = \beta, \beta_1 = \alpha - 1, \gamma_1 = \gamma - 1, \delta_1 = 1 + \beta - \gamma, a_1 = (a\alpha - \lambda)/(\alpha - 1), \epsilon_1 = \alpha$. Alternatively, we refer the reader to [18] where spaces of initial conditions of the sixth Painlevé equation were used to study such confluences. Another component of the vector Y_2 after the gauge transformation satisfies the equation of form (11) with other parameters θ_j, $j = 0, 1, 2$, and another apparent singularity L.

5.2. mc_α

Similar calculations can be done in the case of middle convolution with parameter α. The eigenvalues of matrices $\widetilde{B}_k, k = 1, 2, 3$, in the resulting system (10) are as follows: $1 + \alpha - \gamma, 0; \alpha - \delta, 0; -2 - \beta + \gamma + \delta, 0$. The residue matrix at infinity is the matrix $-(\widetilde{B}_1 + \widetilde{B}_2 + \widetilde{B}_3)$ with eigenvalues $1 - \alpha, \beta - \alpha$. We can diagonalize this matrix by using a gauge transformation for the rank 2 system and write down a scalar equation for the first component of the system (10) after the gauge transformation. It is of form (11) with $\theta_0 = 1 + \alpha - \gamma, \theta_1 = \alpha - \delta, \theta_2 = -2 - \beta + \gamma + \delta$. Also, M and L are parametrized by the accessory parameter of the Heun equation. Eliminating parameter a from these formulae, we get that $M = \epsilon/(\lambda - 1) + (\gamma + \delta - 2 - \beta)/(L - \lambda)$. Substituting L and M into (12) yields condition $(1 + \beta - \gamma)\epsilon = 0$ which in turn leads to $L = 1$. Hence, the apparent singularity coalesces with singularity $t = 1$ and we can choose parameters such that the equation becomes Heun's equation (1).

We remark that in special cases, we get Heun's equation directly for the first component of the vector Y_2 in (10). All arguments also apply for the second component of the vector Y_2. However, it satisfies the equation of form (11) with other parameters and another apparent singularity. Thus, we have demonstrated how equation (11) is used when middle convolution is applied to the Heun hypergeometric system.

6. Concluding remarks

We have studied middle convolution for the Heun hypergeometric system and obtained rank 2 and 3 Fuchsian systems associated with the Heun equation. By repeated application of middle convolution and addition, we can get other systems of a rank greater than 3 by varying the parameter of middle convolution. We hypothesize that by using the hypergeometric systems, one can find transformations (e.g., integral or birational) for other special functions including nonlinear special functions such as the Garnier systems. The simplest example of the sixth Painlevé equation was considered in [7, 8] which seems to support our conjecture, though we did not use the associated hypergeometric system. We plan to address this in more detail in a separate publication.

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